University of Ottawa
MAT 1332D Midterm Exam
Feb. 11, 2009. Duration: 80 Minutes. Instructor: Jing Li

Family Name:______________________

First Name:_______________________

Do **not** write your student ID number on this front page. Please write your student ID number in the space provided on the second page.

Take your time to read the entire paper before you begin to write, and read each question carefully. Remember that certain questions are worth more points than others. Make a note of the questions that you feel confident you can do, and then do those first: <u>you do not</u> have to proceed through the paper in the order given.

- You have 80 minutes to complete this exam.
- This is a closed book exam, and no notes of any kind are allowed. The use of cell phones, pagers or any text storage or communication device **is not permitted.**
- Only the Faculty approved TI-30 calculator is allowed.
- The correct answer requires justification written legibly and logically: you must convince me that you know why your solution is correct. Answer these questions in the space provided. Use the backs of pages if necessary.
- Where it is possible to check your work, do so.
- Good Luck!

Student number:____________________, Total marks: ___________ out of 27 (+2 bonus)

Problem	1	2	3	4	5	6	7
Marks							

Question 1. [3 points] Consider the two functions

$$f(x) = 1 - \frac{1}{x+1}, \qquad g(x) = \frac{1}{2}x.$$

1. (1 point) Show that the functions intersect at the two points $x_1 = 0$ and $x_2 = 1$.
2. (2 points) Find the area enclosed by the two functions between the points of intersection.

Question 1. [3 points] Consider the two functions

$$f(x) = 1 - \frac{2}{x+2}, \qquad g(x) = \frac{1}{3}x.$$

1. (1 point) Show that the functions intersect at the two points $x_1 = 0$ and $x_2 = 1$.
2. (2 points) Find the area enclosed by the two functions between the points of intersection.

Question 1. [3 points] Consider the two functions

$$f(x) = 1 - \frac{3}{x+3}, \qquad g(x) = \frac{1}{4}x.$$

1. (1 point) Show that the functions intersect at the two points $x_1 = 0$ and $x_2 = 1$.
2. (2 points) Find the area enclosed by the two functions between the points of intersection.

Solution version A

$$f(x) = g(x) \quad \Leftrightarrow \quad 1 - \frac{1}{1+x} = \frac{1}{2}x \quad \Leftrightarrow \quad \frac{x}{1+x} = \frac{1}{2}x \quad \Leftrightarrow \quad x^2 - x = 0.$$

The solutions of the quadratic equation are $x_1 = 0$, and $x_2 = 2$. Hence, the functions intersect only at these two points.

The slope of f at zero is $f'(0) = 1$, which is bigger than the slope of g at zero. Hence, the function f is above the function g on the interval $(0, 1)$. (See Figure 1 for illustration)

Then the area between the two graphs is given by

$$\int_0^1 |f(x) - g(x)| dx = \int_0^1 [f(x) - g(x)] dx = \int_0^1 \left[1 - \frac{1}{1+x} - \frac{1}{2}x\right] dx =$$

$$\left[x - \ln|1+x| - \frac{1}{4}x^2\right]\Big|_0^1 = 1 - \ln(2) - 1/4 = 3/4 - \ln(2) \approx 0.0569.$$

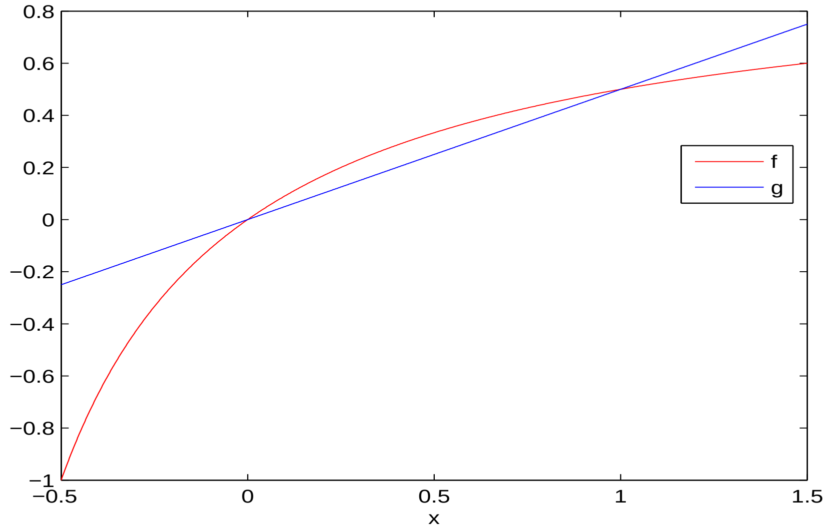

Figure 1: The graphes of function $f(x) = 1 - \frac{1}{x+1}$, and $g(x) = \frac{1}{2}x$.

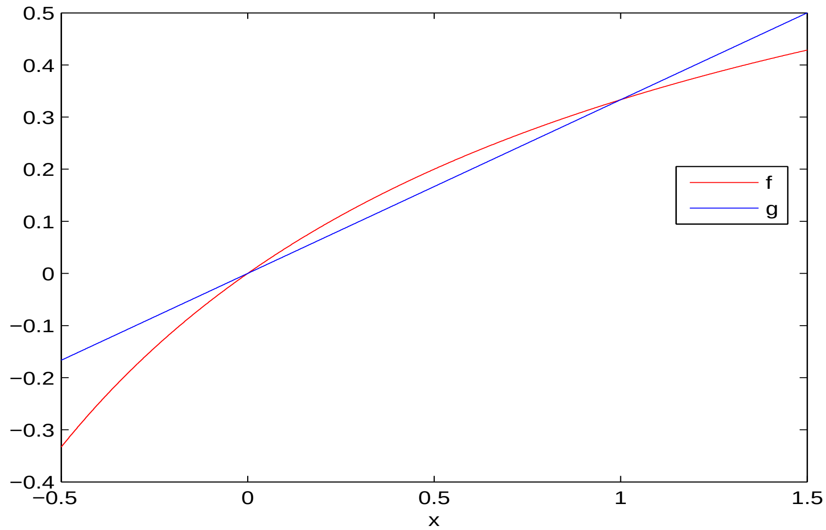

Figure 2: The graphes of function $f(x) = 1 - \frac{2}{x+2}$, and $g(x) = \frac{1}{3}x$.

Solution version B
(See Figure 2 for illustration.)

$$\int_0^1 |f(x) - g(x)| dx = \int_0^1 \left[1 - \frac{2}{2+x} - \frac{1}{3}x\right] dx = 5/6 - 2\ln(3/2) \approx 0.224.$$

Solution version C

(See Figure 3 for illustration.)

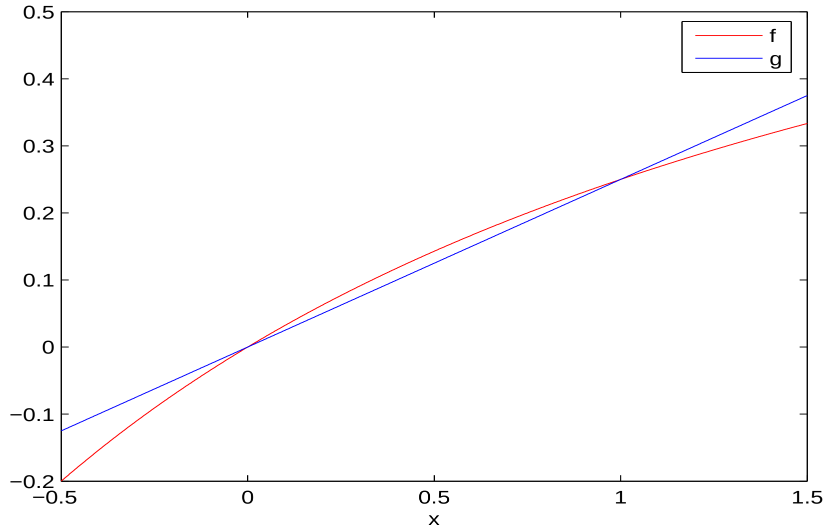

Figure 3: The graphes of function $f(x) = 1 - \frac{3}{x+3}$, and $g(x) = \frac{1}{4}x$.

$$\int_0^1 |f(x) - g(x)| dx = \int_0^1 \left[1 - \frac{3}{3+x} - \frac{1}{4}x\right] dx = 7/8 - 3\ln(4/3) \approx 0.0120.$$

Question 2. [5 points] Suppose that a tree grows in height according to the equation

$$\frac{dH}{dt} = 8e^{-0.4t}, \qquad H(0) = 8.$$

The units are in metres.

1. (2 points) How much does the tree grow between $t = 0$ and $t = 4$?
2. (2 points) Is the growth until $t = \infty$ finite or infinite?
3. (1 point) Will the total height of the tree ever reach 30 m?

Question 2. [5 points] Suppose that a tree grows in height according to the equation

$$\frac{dH}{dt} = 6e^{-0.3t}, \qquad H(0) = 6.$$

The units are in metres.

1. (2 points) How much does the tree grow between $t = 0$ and $t = 4$?
2. (2 points) Is the growth until $t = \infty$ finite or infinite?
3. (1 point) Will the total height of the tree ever reach 30 m?

Question 2. [5 points] Suppose that a tree grows in height according to the equation

$$\frac{dH}{dt} = 7e^{-0.35t}, \qquad H(0) = 7.$$

The units are in metres.

1. (2 points) How much does the tree grow between $t = 0$ and $t = 4$?
2. (2 points) Is the growth until $t = \infty$ finite or infinite?
3. (1 point) Will the total height of the tree ever reach 30 m?

Solution version A

Growth between $t = 0$ and $t = 4$:

$$\int_0^4 \frac{dH}{dt} dt = \int_0^4 8e^{-0.4t} dt = \frac{8}{-0.4} e^{-0.4t}\Big|_0^4 = 20(1 - e^{-1.6}) \approx 15.96.$$

Growth until $t = \infty$:

$$\int_0^\infty \frac{dH}{dt} dt = \lim_{T\to\infty} \int_0^T 8e^{-0.4t} dt = \lim_{T\to\infty} -20e^{-0.4t}\Big|_0^T = \lim_{T\to\infty} 20(1 - e^{-0.4T}) = 20.$$

Since the tree starts at a height of 8 m and grows a maximum of 20 m, it will not reach 30 m.

Solution version B

$$\int_0^4 \frac{dH}{dt}dt = \int_0^4 6e^{-0.3t}dt = 20(1 - e^{-1.2}) \approx 13.98.$$

Growth until $t = \infty$:

$$\int_0^\infty \frac{dH}{dt}dt = \lim_{T\to\infty} \int_0^T 6e^{-0.3t}dt = \lim_{T\to\infty} -20e^{-0.3t}\Big|_0^T = \lim_{T\to\infty} 20(1 - e^{-0.3T}) = 20.$$

Since the tree starts at a height of 6 m and grows a maximum of 20 m, it will not reach 30 m.

Solution version C

$$\int_0^4 \frac{dH}{dt}dt = \int_0^4 7e^{-0.35t}dt = 20(1 - e^{-1.4}) \approx 15.07.$$

Growth until $t = \infty$:

$$\int_0^\infty \frac{dH}{dt}dt = \lim_{T\to\infty} \int_0^T 7e^{-0.35t}dt = \lim_{T\to\infty} -20e^{-0.35t}\Big|_0^T = \lim_{T\to\infty} 20(1 - e^{-0.35T}) = 20.$$

Since the tree starts at a height of 7 m and grows a maximum of 20 m, it will not reach 30 m.

Question 3. [4 points] Find the indefinite integral

$$\int \frac{5x-1}{x^2+x-12}dx.$$

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$$\int \frac{5x+1}{x^2-x-12}dx.$$

Question 3. [4 points] Find the indefinite integral

$$\int \frac{5x+2}{x^2-x-20}dx.$$

Solution version A

The denominator factors as $x^2+x-12=(x-3)(x+4)$, hence we have two distinct roots. Then we use partial fractions

$$\frac{5x-1}{x^2+x-12}=\frac{A}{x-3}+\frac{B}{x+4}=\frac{A(x+4)+B(x-3)}{(x-3)(x+4)}=\frac{(A+B)x+(4A-3B)}{(x-3)(x+4)}.$$

Then comparing coefficients we have $A+B=5, 4A-3B=-1$, thus we get $A=2, B=3$. Then we can integrate

$$\int \frac{5x-1}{x^2+x-12}dx=\int \left[\frac{2}{x-3}+\frac{3}{x+4}\right]dx=2\ln|x-3|+3\ln|x+4|+C.$$

Solution version B

The denominator factors as $x^2-x-12=(x-4)(x+3)$, hence we have two distinct roots. Then we use partial fractions

$$\frac{5x+1}{x^2-x-12}=\frac{A}{x-4}+\frac{B}{x+3}=\frac{A(x+3)+B(x-4)}{(x-4)(x+3)}=\frac{(A+B)x+(3A-4B)}{(x-4)(x+3)},$$

Then comparing coefficients we have $A+B=5, 3A-4B=1$, thus we get $A=3, B=2$. Then we can integrate

$$\int \frac{5x+1}{x^2-x-12}dx=\int \left[\frac{3}{x-4}+\frac{2}{x+3}\right]dx=3\ln|x-4|+2\ln|x+3|+C.$$

Solution version C

The denominator factors as $x^2-x-20=(x-5)(x+4)$, hence we have two distinct roots. Then we use partial fractions

$$\frac{5x+2}{x^2-x-20}=\frac{A}{x-5}+\frac{B}{x+4}=\frac{A(x+4)+B(x-5)}{(x-5)(x+4)}=\frac{(A+B)x+(4A-5B)}{(x-5)(x+4)},$$

Then comparing coefficients we have $A+B=5, 4A-5B=2$, thus we get $A=3, B=2$. Then we can integrate

$$\int \frac{5x+2}{x^2-x-20}dx=\int \left[\frac{3}{x-5}+\frac{2}{x+4}\right]dx=3\ln|x-5|+2\ln|x+4|+C.$$

Question 4. [2 points] Does the following improper integral converge or diverge? If it converges, give its value.

$$\int_1^2 \frac{3}{(x-1)^{2/3}} dx.$$

Question 4. [2 points] Does the following improper integral converge or diverge? If it converges, give its value.

$$\int_1^2 \frac{5}{(x-1)^{2/3}} dx.$$

Question 4. [2 points] Does the following improper integral converge or diverge? If it converges, give its value.

$$\int_1^2 \frac{7}{(x-1)^{2/3}} dx.$$

Solution version A

The denominator is zero for $x = 1$. We substitute $u = x - 1$ and use the definition for type II improper integrals to get

$$\int_1^2 \frac{3}{(x-1)^{2/3}} dx = \int_0^1 3u^{-\frac{2}{3}} du = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 3u^{-\frac{2}{3}} du = \lim_{\varepsilon \to 0^+} 3 \cdot 3u^{\frac{1}{3}}|_\varepsilon^1 = \lim_{\epsilon \to 0} 9(1 - \varepsilon^{\frac{1}{3}}) = 9.$$

Hence, the integral converges and its value is 9.

Solution version B

The denominator is zero for $x = 1$. We substitute $u = x - 1$ and use the definition for type II improper integrals to get

$$\int_1^2 \frac{5}{(x-1)^{2/3}} dx = \int_0^1 5u^{-\frac{2}{3}} du = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 5u^{-\frac{2}{3}} du = \lim_{\varepsilon \to 0^+} 5 \cdot 3u^{\frac{1}{3}}|_\varepsilon^1 = \lim_{\epsilon \to 0} 15(1 - \varepsilon^{\frac{1}{3}}) = 15.$$

Hence, the integral converges and its value is 15.

Solution version C

The denominator is zero for $x = 1$. We substitute $u = x - 1$ and use the definition for type II improper integrals to get

$$\int_1^2 \frac{7}{(x-1)^{2/3}} dx = \int_0^1 7u^{-\frac{2}{3}} du = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 7u^{-\frac{2}{3}} du = \lim_{\varepsilon \to 0^+} 7 \cdot 3u^{\frac{1}{3}}|_\varepsilon^1 = \lim_{\epsilon \to 0} 21(1 - \varepsilon^{\frac{1}{3}}) = 21.$$

Hence, the integral converges and its value is 21.

Question 5. [6 points] An intravenous drip is the continuous infusion of fluids into the human blood stream, which is often performed in hospitals to correct electrolyte balance. We denote by M the mass (in milligrams) of electrolytes that enter the bloodstream per unit time. These electrolytes are taken up by the body at a rate k. We denote the concentration of electrolytes in the blood stream by E (in milligrams per litre). The the equation for E is

$$\frac{dE}{dt} = aM - kE,$$

where a converts mass of electrolytes into concentration in the blood stream, i.e. $1/a$ is the amount of blood of the person. Choose the parameter values: $a = 6, k = 1/5, M = 10$.

1. (2 points) Find the steady state E^*. Determine its stability, using the derivative test. What is the concentration in the blood stream in the long run?
2. (2 points) Draw the phase line diagram and sketch the solution curve $E(t)$, starting at $E(0) = 60$.
3. (**Bonus 2 points**) Find an explicit solution to the equation with initial value $E(0) = 60$. [Hint: from the phase line diagram, you know that the solution satisfies $E(t) < 300$ for all $t > 0$.]

Question 5. [6 points] An intravenous drip is the continuous infusion of fluids into the human blood stream, which is often performed in hospitals to correct electrolyte balance. We denote by M the mass (in milligrams) of electrolytes that enter the bloodstream per unit time. These electrolytes are taken up by the body at a rate k. We denote the concentration of electrolytes in the blood stream by E (in milligrams per litre). The the equation for E is

$$\frac{dE}{dt} = aM - kE,$$

where a converts mass of electrolytes into concentration in the blood stream, i.e. $1/a$ is the amount of blood of the person. Choose the parameter values: $a = 5, k = 1/4, M = 15$.

1. (2 points) Find the steady state E^*. Determine its stability, using the derivative test. What is the concentration in the blood stream in the long run?
2. (2 points) Draw the phase line diagram and sketch the solution curve $E(t)$, starting at $E(0) = 50$.
3. (**Bonus 2 points**) Find an explicit solution to the equation with initial value $E(0) = 50$. [Hint: from the phase line diagram, you know that the solution satisfies $E(t) < 300$ for all $t > 0$.]

Question 5. [6 points] An intravenous drip is the continuous infusion of fluids into the human blood stream, which is often performed in hospitals to correct electrolyte balance. We denote by M the mass (in milligrams) of electrolytes that enter the bloodstream per unit time. These electrolytes are taken up by the body at a rate k. We denote the concentration of electrolytes in the blood stream by E (in milligrams per litre). The the equation for E is

$$\frac{dE}{dt} = aM - kE,$$

where a converts mass of electrolytes into concentration in the blood stream, i.e. $1/a$ is the amount of blood of the person. Choose the parameter values: $a = 4, k = 1/3, M = 20$.

1. (2 points) Find the steady state E^*. Determine its stability, using the derivative test. What is the concentration in the blood stream in the long run?

2. (2 points) Draw the phase line diagram and sketch the solution curve $E(t)$, starting at $E(0) = 40$.

3. (**Bonus 2 points**) Find an explicit solution to the equation with initial value $E(0) = 40$. [Hint: from the phase line diagram, you know that the solution satisfies $E(t) < 240$ for all $t > 0$.]

Solution version A

Part 1: We denote the right-hand side of the differential equation by $f(E) = aM - kE$. The steady state is given by the equation $f(E^*) = 0$. Hence,

$$aM - kE^* = 0 \quad \Leftrightarrow \quad E^* = \frac{aM}{k} = \frac{6 \times 10}{\frac{1}{5}} = 300.$$

To check stability, we have to evaluate the derivative of f at the steady state.

$$f'(E) = -k = -\frac{1}{5} < 0.$$

Hence, the derivative is negative (independent of the steady state). Then the steady state is stable. In the long run, the concentration will approach the steady state value of

$$E^* = \frac{aM}{k} = \frac{6 \times 10}{\frac{1}{5}} = 300.$$

Part 2: The phase-line diagram and the solution curve are shown in Figure 4 and Figure 5.

Part 3: For the bonus question, we separate variables and integrate

$$\int \frac{dE}{aM - kE} = \int dt \quad \Rightarrow \quad \frac{-1}{k} \ln|aM - kE| = t + C$$

We take exponentials on both sides to get

$$|aM - kE| = De^{-kt}, \qquad \text{and} \quad D = e^{-kC} \text{ is a constant.}$$

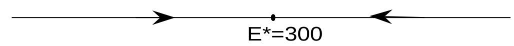

Figure 4: The phase line diagram.

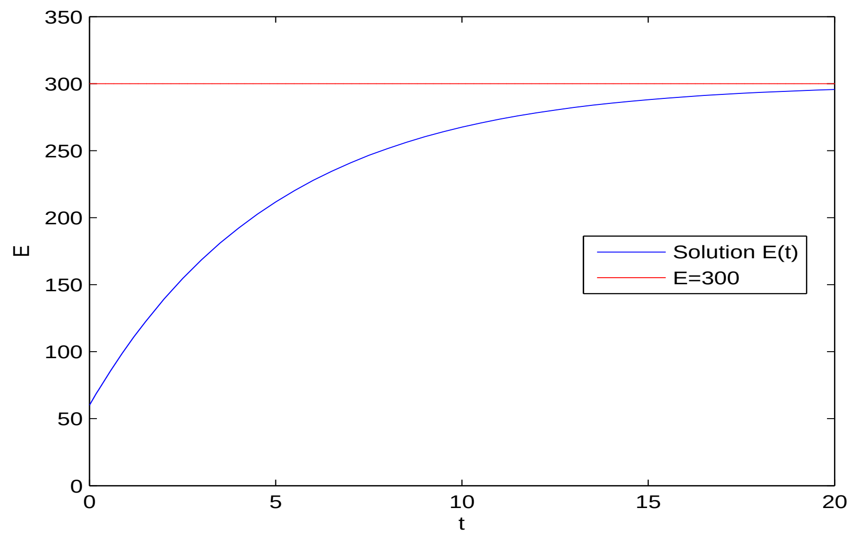

Figure 5: The solution curve $E(t)$ starting at $E(0) = 60$.

Since the solution starts between 0 and aM/k, it will remain between 0 and aM/k, so that we can drop the absolute value bars. Then we can solve for E to get

$$E(t) = \frac{1}{k}(aM - De^{-kt}).$$

The constant is given by the initial value as

$$E(0) = \frac{aM - D}{k}, \quad \text{or} \quad D = aM - kE(0) = 6 \times 10 - \frac{1}{5} \times 60 = 48.$$

Hence the solution is

$$E(t) = \frac{1}{k}(aM - De^{-kt}) = \frac{1}{1/5}(6 \times 10 - 48 \times e^{-\frac{1}{5}t}) = 300 - 240e^{-\frac{1}{5}t}.$$

Solution version B

Part 1: We denote the right-hand side of the differential equation by $f(E) = aM - kE$. The steady state is given by the equation $f(E^*) = 0$. Hence,

$$aM - kE^* = 0 \quad \Leftrightarrow \quad E^* = \frac{aM}{k} = \frac{5 \times 15}{\frac{1}{4}} = 300.$$

To check stability, we have to evaluate the derivative of f at the steady state.

$$f'(E) = -k = -\frac{1}{4} < 0.$$

Hence, the derivative is negative (independent of the steady state). Then the steady state is stable. In the long run, the concentration will approach the steady state value of

$$E^* = \frac{aM}{k} = \frac{5 \times 15}{\frac{1}{4}} = 300.$$

Part 2: The phase-line diagram and the solution curve are shown in Figure 6 and Figure 7.

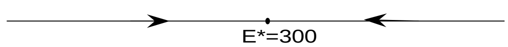

Figure 6: The phase line diagram.

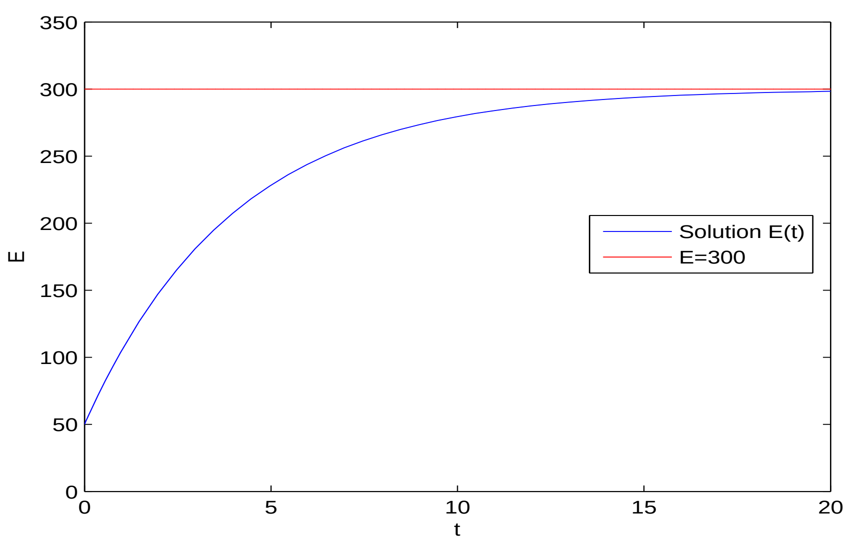

Figure 7: The solution curve $E(t)$ starting at $E(0) = 50$.

Part 3: For the bonus question, we separate variables and integrate

$$\int \frac{dE}{aM - kE} = \int dt \quad \Rightarrow \quad \frac{-1}{k} \ln |aM - kE| = t + C$$

We take exponentials on both sides to get

$$|aM - kE| = De^{-kt}, \qquad \text{and} \quad D = e^{-kC} \text{ is a constant.}$$

Since the solution starts between 0 and aM/k, it will remain between 0 and aM/k, so that we can drop the absolute value bars. Then we can solve for E to get

$$E(t) = \frac{1}{k}(aM - De^{-kt}).$$

The constant is given by the initial value as

$$E(0) = \frac{aM - D}{k}, \quad \text{or} \quad D = aM - kE(0) = 5 \times 15 - \frac{1}{4} \times 50 = \frac{250}{4}.$$

Hence the solution is

$$E(t) = \frac{1}{k}(aM - De^{-kt}) = \frac{1}{1/4}(5 \times 15 - \frac{250}{4} \times e^{-\frac{1}{4}t}) = 300 - 250e^{-\frac{1}{4}t}.$$

Solution version C

Part 1: We denote the right-hand side of the differential equation by $f(E) = aM - kE$. The steady state is given by the equation $f(E^*) = 0$. Hence,

$$aM - kE^* = 0 \quad \Leftrightarrow \quad E^* = \frac{aM}{k} = \frac{4 \times 20}{\frac{1}{3}} = 240.$$

To check stability, we have to evaluate the derivative of f at the steady state.

$$f'(E) = -k = -\frac{1}{3} < 0.$$

Hence, the derivative is negative (independent of the steady state). Then the steady state is stable. In the long run, the concentration will approach the steady state value of

$$E^* = \frac{aM}{k} = \frac{4 \times 20}{\frac{1}{3}} = 240.$$

Part 2: The phase-line diagram and the solution curve are shown in Figure 8 and Figure 9.

Part 3: For the bonus question, we separate variables and integrate

$$\int \frac{dE}{aM - kE} = \int dt \quad \Rightarrow \quad \frac{-1}{k} \ln |aM - kE| = t + C$$

We take exponentials on both sides to get

$$|aM - kE| = De^{-kt}, \qquad \text{and} \quad D = e^{-kC} \text{ is a constant.}$$

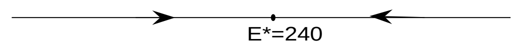

Figure 8: The phase line diagram.

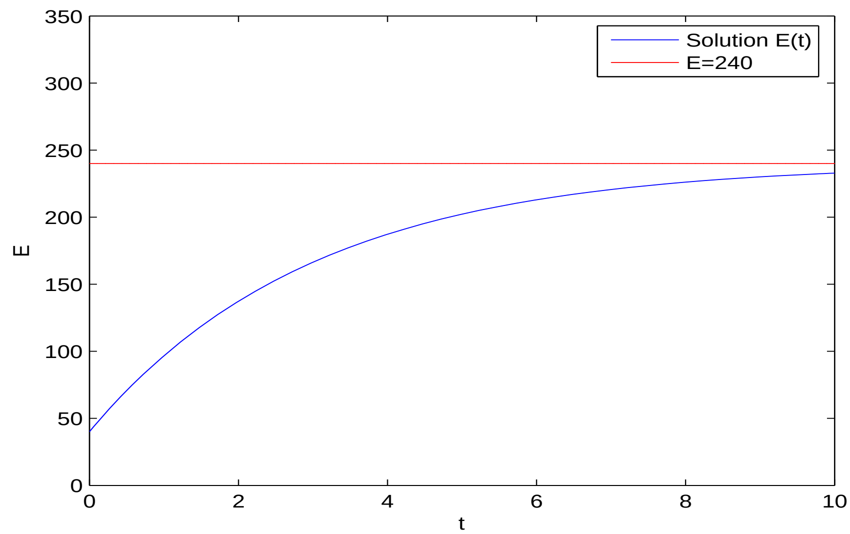

Figure 9: The solution curve $E(t)$ starting at $E(0) = 40$.

Since the solution starts between 0 and aM/k, it will remain between 0 and aM/k, so that we can drop the absolute value bars. Then we can solve for E to get

$$E(t) = \frac{1}{k}(aM - De^{-kt}).$$

The constant is given by the initial value as

$$E(0) = \frac{aM - D}{k}, \quad \text{or} \quad D = aM - kE(0) = 4 \times 20 - \frac{1}{3} \times 40 = \frac{200}{3}.$$

Hence the solution is

$$E(t) = \frac{1}{k}(aM - De^{-kt}) = \frac{1}{1/3}(4 \times 20 - \frac{200}{3} \times e^{-\frac{1}{3}t}) = 2400 - 200e^{-\frac{1}{3}t}.$$

Question 6. [3 points] Solve the separable differential equation

$$\frac{db}{dt} = \frac{1}{25+t^2}b, \qquad b(0) = \frac{1}{5}.$$

Question 6. [3 points] Solve the separable differential equation

$$\frac{db}{dt} = \frac{1}{64+t^2}b, \qquad b(0) = \frac{1}{8}.$$

Question 6. [3 points] Solve the separable differential equation

$$\frac{db}{dt} = \frac{1}{100+t^2}b, \qquad b(0) = \frac{1}{10}.$$

Solution version A

Separating variables, we get

$$\frac{db}{b} = \frac{1}{25+t^2}dt.$$

Integration on both sides yields

$$\ln|b| = \frac{1}{5}\arctan(\frac{t}{5}) + C.$$

Taking exponents on both sides, we get

$$b(t) = Ke^{\frac{1}{5}\arctan(\frac{t}{5})}, \quad K = e^C.$$

Using the initial value, we see that $\arctan(0) = 0$ and $b(0) = K$ so that the solution is

$$b(t) = \frac{1}{5}e^{\frac{1}{5}\arctan(\frac{t}{5})}.$$

Solution version B

Separating variables, we get

$$\frac{db}{b} = \frac{1}{64+t^2}dt.$$

Integration on both sides yields

$$\ln|b| = \frac{1}{8}\arctan(\frac{t}{8}) + C.$$

Taking exponents on both sides, we get

$$b(t) = Ke^{\frac{1}{8}\arctan(\frac{t}{8})}, \quad K = e^C.$$

Using the initial value, we see that $\arctan(0) = 0$ and $b(0) = K$ so that the solution is

$$b(t) = \frac{1}{8} e^{\frac{1}{8}\arctan(\frac{t}{8})}.$$

Solution version C

Separating variables, we get

$$\frac{db}{b} = \frac{1}{100 + t^2} dt.$$

Integration on both sides yields

$$\ln|b| = \frac{1}{10}\arctan(\frac{t}{10}) + C.$$

Taking exponents on both sides, we get

$$b(t) = Ke^{\frac{1}{10}\arctan(\frac{t}{10})}, \quad K = e^C.$$

Using the initial value, we see that $\arctan(0) = 0$ and $b(0) = K$ so that the solution is

$$b(t) = \frac{1}{10} e^{\frac{1}{10}\arctan(\frac{t}{10})}.$$

Question 7. [6 points] Consider the autonomous differential equation

$$\frac{dx}{dt} = x(x^2 - 5x + 4).$$

Do not solve this equation explicitly!

1. (1 point) Find the steady states x_1^*, x_2^*, x_3^*.
2. (2 points) Find the stability of x_1^*, x_2^*, x_3^*, using the derivative test.
3. (1 point) Draw the phase line diagram.
4. (2 points) Without solving the equation explicitly, sketch the graph of $x(t)$ starting at $x(0) = 3.5$. Indicate inflections points if there are any.

Question 7. [6 points] Consider the autonomous differential equation

$$\frac{dx}{dt} = x(x^2 - 6x + 8).$$

Do not solve this equation explicitly!

1. (1 point) Find the steady states x_1^*, x_2^*, x_3^*.
2. (2 points) Find the stability of x_1^*, x_2^*, x_3^*, using the derivative test.
3. (1 point) Draw the phase line diagram.
4. (2 points) Without solving the equation explicitly, sketch the graph of $x(t)$ starting at $x(0) = 0.25$. Indicate inflections points if there are any.

Question 7. [6 points] Consider the autonomous differential equation

$$\frac{dx}{dt} = x(x^2 - 7x + 12).$$

Do not solve this equation explicitly!

1. (1 point) Find the steady states x_1^*, x_2^*, x_3^*.
2. (2 points) Find the stability of x_1^*, x_2^*, x_3^*, using the derivative test.
3. (1 point) Draw the phase line diagram.
4. (2 points) Without solving the equation explicitly, sketch the graph of $x(t)$ starting at $x(0) = 0.5$. Indicate inflections points if there are any.

Solution version A

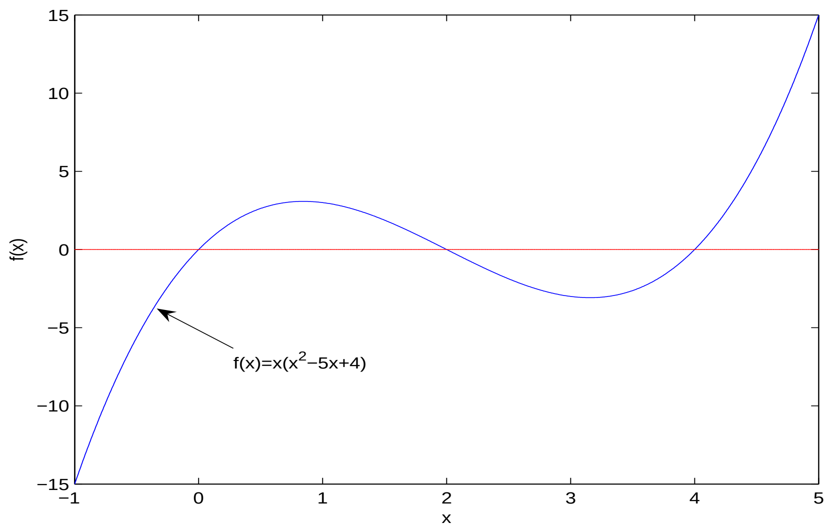

Figure 10: The graph of function $f(x) = x(x^2 - 5x + 4)$.

Part 1: We denote $f(x) = x(x^2 - 5x + 4)$ (see the graph of $f(x)$ for illustration in Figure 10). The zeros of f are the steady states of the differential equation, i.e.,

$$\begin{aligned} x(x^2 - 5x + 4) &= 0, \\ x(x-1)(x-4) &= 0, \\ x_1^* = 0, \quad x_2^* = 1, \quad x_3^* &= 4. \end{aligned}$$

Part 2: The derivative of $f = x^3 - 5x^2 + 4x$ is

$$f'(x) = 3x^2 - 10x + 4.$$

And then

$$\begin{aligned} &f'(0) = 4 > 0, \quad \text{hence} \quad x_1^* \quad \text{is unstable;} \\ &f'(1) = -3 < 0, \quad \text{hence} \quad x_2^* \quad \text{is stable;} \\ &f'(4) = 12 > 0, \quad \text{hence} \quad x_3^* \quad \text{is unstable.} \end{aligned}$$

Part 3: The phase line diagram is given in Figure 11.

Part 4: The solution curve is shown in Figure 12.

To indicate the inflection points of the solution curve, we need to check the roots of $f'(x) = 3x^2 - 10x + 4$, i.e.,

$$x_{1,2} = \frac{10 \pm \sqrt{10^2 - 4 \times 3 \times 4}}{2 \times 3} = \frac{5 \pm \sqrt{13}}{3}.$$

Noting that $\frac{5+\sqrt{13}}{3} < 3.5$, there is one inflection point $x = \frac{5+\sqrt{13}}{3}$ in the solution curve of $x(t)$ starting at $x(0) = 3.5$.

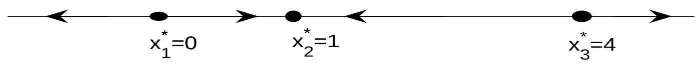

Figure 11: The phase line diagram of the differential equation $\frac{dx}{dt} = x(x^2 - 5x + 4)$.

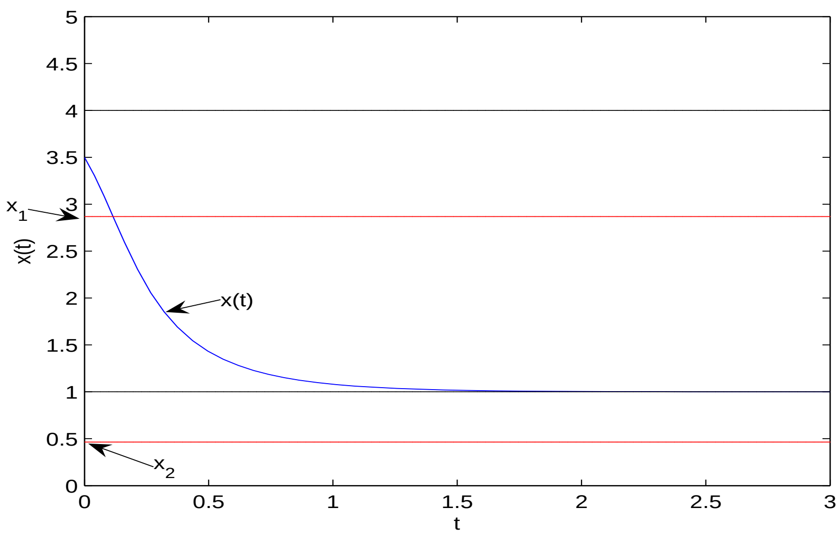

Figure 12: The solution curve of $x(t)$ starting at $x(0) = 3.5$. Infection points on the red lines $x_1 = \frac{5+\sqrt{13}}{3}$ and $x_2 = \frac{5-\sqrt{13}}{3}$.

Solution version B

Part 1: We denote $f(x) = x(x^2 - 6x + 8)$ (see the graph of $f(x)$ for illustration in Figure 13). The zeros of f are the steady states of the differential equation, i.e.,

$$\begin{aligned} x(x^2 - 6x + 8) &= 0, \\ x(x-2)(x-4) &= 0, \\ x_1^* = 0, \quad x_2^* = 2, \quad x_3^* &= 4. \end{aligned}$$

Part 2: The derivative of $f = x^3 - 6x^2 + 8x$ is

$$f'(x) = 3x^2 - 12x + 8.$$

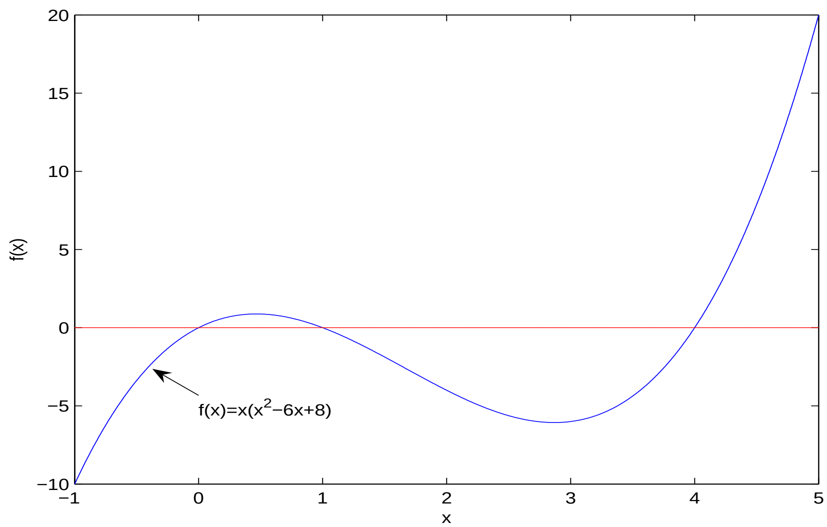

Figure 13: The graph of function $f(x) = x(x^2 - 6x + 8)$.

And then

$$\begin{aligned} f'(0) &= 8 > 0, \quad \text{hence} \quad x_1^* \quad \text{is unstable;} \\ f'(2) &= -4 < 0, \quad \text{hence} \quad x_2^* \quad \text{is stable;} \\ f'(4) &= 8 > 0, \quad \text{hence} \quad x_3^* \quad \text{is unstable.} \end{aligned}$$

Part 3: The phase line diagram is given in Figure 14.

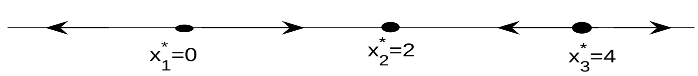

Figure 14: The phase line diagram of the differential equation $\frac{dx}{dt} = x(x^2 - 6x + 8)$.

Part 4: The solution curve is shown in Figure 15.

To indicate the inflection points of the solution curve, we need to check the roots of $f'(x) = 3x^2 - 12x + 8$, i.e.,

$$x_{1,2} = \frac{12 \pm \sqrt{12^2 - 4 \times 3 \times 8}}{2 \times 3} = 2 \pm \frac{2\sqrt{3}}{3}.$$

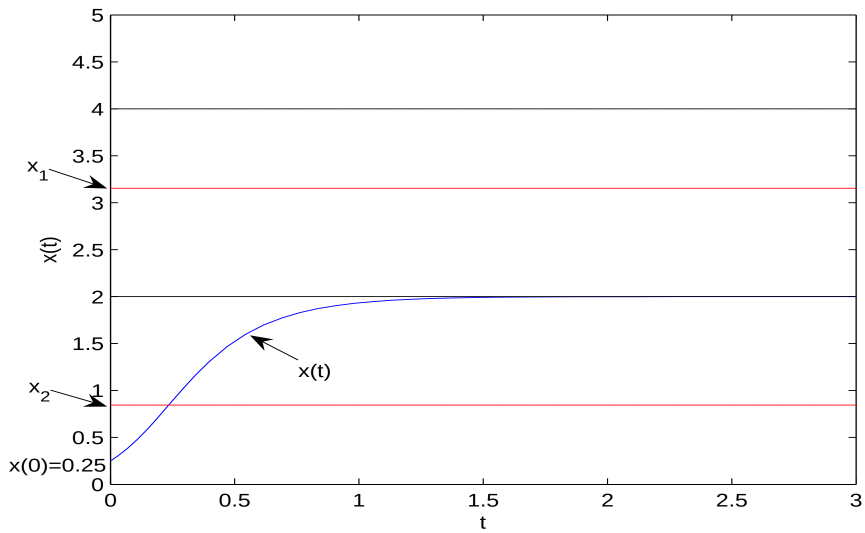

Figure 15: The solution curve of $x(t)$ starting at $x(0) = 0.25$. Infection points on the red lines $x_1 = 2 + \frac{2\sqrt{3}}{3}$ and $x_2 = 2 - \frac{2\sqrt{3}}{3}$.

Noting that $2 - \frac{2\sqrt{3}}{3} > 0.25$, there is one inflection point $x = 2 - \frac{2\sqrt{3}}{3}$ in the solution curve of $x(t)$ starting at $x(0) = 0.25$.

Solution version C

Part 1: We denote $f(x) = x(x^2 - 7x + 12)$ (see the graph of $f(x)$ for illustration in Figure 13). The zeros of f are the steady states of the differential equation, i.e.,

$$\begin{aligned} x(x^2 - 7x + 12) &= 0, \\ x(x-3)(x-4) &= 0, \\ x_1^* = 0, \quad x_2^* = 3, \quad x_3^* &= 4. \end{aligned}$$

Part 2: The derivative of $f = x^3 - 7x^2 + 12x$ is

$$f'(x) = 3x^2 - 14x + 12.$$

And then

$$\begin{aligned} &f'(0) = 12 > 0, \quad \text{hence} \quad x_1^* \quad \text{is unstable;} \\ &f'(3) = -3 < 0, \quad \text{hence} \quad x_2^* \quad \text{is stable;} \\ &f'(4) = 4 > 0, \quad \text{hence} \quad x_3^* \quad \text{is unstable.} \end{aligned}$$

Part 3: The phase line diagram is given in Figure 17.

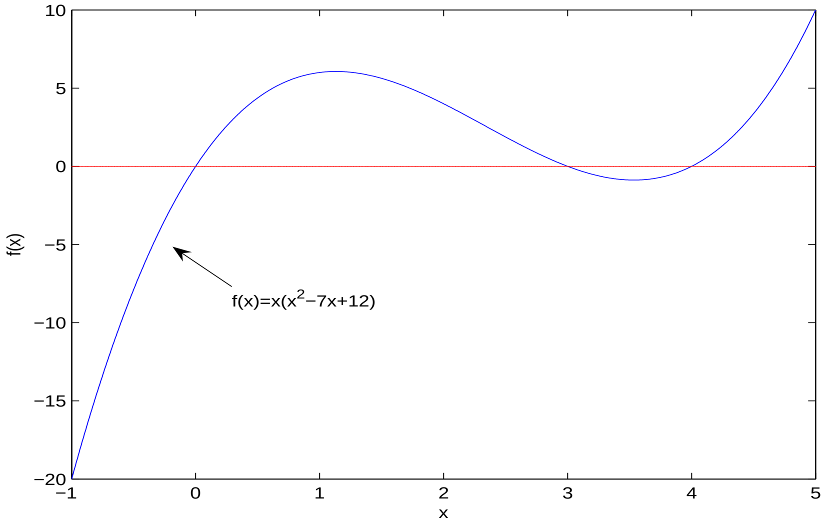

Figure 16: The graph of function $f(x) = x(x^2 - 7x + 12)$.

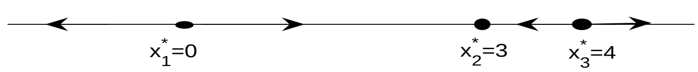

Figure 17: The phase line diagram of the differential equation $\frac{dx}{dt} = x(x^2 - 7x + 12)$.

Part 4: The solution curve is shown in Figure 18.

To indicate the inflection points of the solution curve, we need to check the roots of $f'(x) = 3x^2 - 14x + 12$, i.e.,

$$x_{1,2} = \frac{14 \pm \sqrt{14^2 - 4 \times 3 \times 12}}{2 \times 3} = \frac{7 \pm \sqrt{13}}{3}.$$

Noting that $\frac{7-\sqrt{13}}{3} > 0.5$, there is one inflection point in the solution curve of $x(t)$ starting at $x(0) = 0.5$.

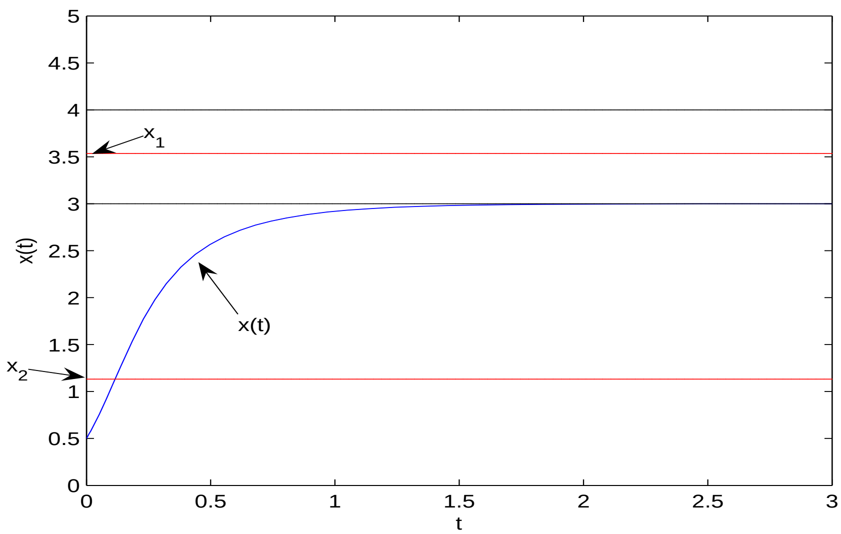

Figure 18: The solution curve of $x(t)$ starting at $x(0) = 0.5$. Infection points on the red lines $x_1 = \frac{7+\sqrt{13}}{3}$ and $x_2 = \frac{7-\sqrt{13}}{3}$.